MMAT5030 Notes 8

1. Fourier Transform

Fourier series expansion in general works for periodic functions. For functions defined on the real line possessing some decay properties (so they are not periodic) the analogous expansion is called the Fourier transform of the function. Previously we discussed Fourier series from the mapping point of view. It is advantageous to recall it. Indeed, we set

$$R_{2\pi} = \{\text{All } 2\pi\text{-periodic, Riemann integrable complex-valued functions}\}\,,$$

$$C^\infty_{2\pi} = \{\text{ All } 2\pi\text{-periodic, infinitely differentiable functions}\}\,,$$
$$\mathcal{S}_0 = \{\{c_n\} : \ c_n \to 0, \text{as } n \to \pm\infty\}\,,$$

and

$$\mathcal{S}_{rd} = \{\{c_n\} \in \mathcal{S}_0 : \ \{c_n\} \text{ is rapidly decreasing }\}\,.$$

It is more convenient to consider complex-valued functions when coming to Fourier transform. Given $f \in R_{2\pi}$, its "Fourier transform" Φ is

$$\Phi(f) = \{c_n\}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, n \in \mathbb{Z}\ .$$

The Fourier transform Φ establishes a map from the space $R_{2\pi}$ to $\mathcal{S}_0$, as a result of Riemann-Lebesgue Lemma. On the other hand, the inverse Fourier transform Ψ is given by

$$\Psi(\{c_n\}) = \sum_{n=-\infty}^{\infty} c_n e^{inx}\ .$$

However, due to the difficulty of convergence, Ψ is not well-defined for every bisequences in $\mathcal{S}_0$. In order to obtain a one-to-one correspondence, we restrict our attention to the subspace $\mathcal{S}_{rd}$ consisting of rapidly decreasing bisequences so that $\Psi(\{c_n\})$ is always well-defined. Indeed, we showed that

$$\Psi \circ \Phi = Id \text{ on } C^\infty_{2\pi}, \text{ and } \Phi \circ \Psi = Id \text{ on } \mathcal{S}_{rd}.$$

Therefore, the Fourier transform sets up a one-to-one correspondence between $C^\infty_{2\pi}$ and $\mathcal{S}_{rd}$.

Now, let us consider functions defined on $\mathbb{R}$, that is, the whole real line. We call a function f **absolutely Riemann integrable** if it is (improperly) Riemann integrable on any $[a,b]$ and both

$$\lim_{a\to-\infty}\int_a^0 |f(x)|dx$$

and

$$\lim_{b\to\infty}\int_0^b |f(x)|dx$$

exist. For such f we set

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx\ .$$

We will denote the class of all absolutely Riemann integrable functions by $R(\mathbb{R})$ or $R(-\infty,\infty)$. Clearly it carries the structure of a vector space over $\mathbb{C}$.

For every absolutely integrable function f on $\mathbb{R}$, we define its **Fourier transform** to be

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\ .$$

Usually we use $\widehat{f}$ or $\widehat{f}(\xi)$ to denote $\mathcal{F}(f)$. The new variable ξ is x in Greek.

Proposition 8.1. For $f\in R(\mathbb{R})$, $\widehat{f}$ is continuous on $\mathbb{R}$ and satisfies

$$\lim_{|\xi|\to\infty}\widehat{f}(\xi) = 0\ .$$

The decay of $\widehat{f}$ at infinity is the analog of the Riemann-Lebesgue Lemma.

Proof. Let $f = f_1 + if_2$ where f_1 and f_2 are respectively the real and imaginary parts of f. We have

$$\begin{aligned}\widehat{f}(\xi) &= \int_{-\infty}^{\infty} f_1(x)\cos\xi x dx + \int_{-\infty}^{\infty} f_2(x)\sin\xi x dx\\ &\quad -i\int_{-\infty}^{\infty} f_1(x)\sin\xi x dx + i\int_{-\infty}^{\infty} f_2(x)\cos\xi x dx\ .\end{aligned}$$

Let us show that the first integral on the right hand side is continuous in ξ. For $\xi_0\in\mathbb{R}$, we fix a large M such that

$$\int_{|x|\geq M}|f_1(x)|dx < \frac{\varepsilon}{4}\ .$$

Then

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f_1(x)\cos\xi x dx - \int_{-\infty}^{\infty} f_1(x)\cos\xi_0 x dx\right| &\leq \left|\int_{-M}^{M} f_1(x)(\cos\xi x-\cos\xi_0 x)dx\right|\\ &\quad + \left|\int_{|x|\geq M} f_1(x)(\cos\xi x-\cos\xi_0 x)dx\right|\\ &\leq \left|\int_{-M}^{M} f_1(x)(\cos\xi x-\cos\xi_0 x)dx\right| +\\ &\quad 2\left|\int_{|x|\geq M} f_1(x)dx\right|\\ &< \left|\int_{-M}^{M} f_1(x)(\cos\xi x-\cos\xi_0 x)dx\right| + \frac{\varepsilon}{2}\ .\end{aligned}$$

By Mean-Value Theorem, we have

$$\cos\xi x - \cos\xi_0 x = -(\sin\xi' x)(\xi-\xi_0)x \ ,$$

where ξ' lies between ξ and ξ_0. It follows that

$$\left|\int_{|x|\le M} f_1(x)(\cos\xi x - \cos\xi_0 x)dx\right| \le M|\xi-\xi_0|\int_{|x|<M}|f_1(x)|dx \ .$$

We can find a small $\delta>0$ such that Whenever $|\xi-\xi_0|<\delta$,

$$\left|\int_{|x|\ge M} f_1(x)(\cos\xi x - \cos\xi_0 x)dx\right| < \frac{\varepsilon}{2} \ .$$

Putting things together, we have shown that the first integral term in $\widehat{f}$ is continuous everywhere. By the same reasoning the other three integrals are continuous.

To prove the decay property, again we examine the first integral and fix the same large M above. Then

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f_1(x)\cos\xi x dx\right| &\le \left|\int_{-M}^{M} f_1(x)\cos\xi x dx\right| + \left|\int_{|x|\ge M} f_1(x)\cos\xi x dx\right| \\ &\le \left|\int_{-M}^{M} f_1(x)\cos\xi x dx\right| + \frac{\varepsilon}{4} \ .\end{aligned}$$

Following the proof of Riemann-Lebesgue Lemma (simply replacing n by ξ), one can show that the integral above tends to 0 as $\xi\to\infty$. Similarly we can treat the other three integrals. $\square$

This proposition suggests the image of Fourier transform should be taken to be the space

$$C_0(\mathbb{R}) \equiv \left\{\varphi\in C(\mathbb{R}) : \lim_{\xi\to\pm\infty}\varphi(\xi)=0\right\} \ .$$

We note that $R(\mathbb{R})$ and $C_0(\mathbb{R})$ correspond to $R_{2\pi}$ and $\mathcal{S}_0$ in Fourier series.

To find the inverse Fourier transform a formal argument (see next section) suggests that it is given by

$$\mathcal{G}(\varphi)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\xi)e^{i\xi x}d\xi \ . \tag{1}$$

It is not hard to see that this map is well-defined when φ is absolutely integrable. Unfortunately, it is not clear and actually not true that it is well-defined for functions in $C_0(\mathbb{R})$. For instance, the function $(1+x^2)^{-1/2}$ belongs to $C_0(\mathbb{R})$ but not in $R(\mathbb{R})$. Thus we are in the same situation of Fourier series. We need to restrict the classes of functions in order to define the inverse Fourier transform. To this end we introduce the Schwartz class of functions. A (complex-valued) function is called **rapidly decreasing** if it is infinitely differentiable and satisfies, for each k, j,

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(j)}(x)| < \infty.$$

Examples of rapidly decreasing functions include all infinitely differentiable function which vanish outside some bounded interval and notably the function $x \mapsto e^{-ax^2}, a > 0$. The collection of all rapidly decreasing functions form a vector space called **Schwartz space** $\mathcal{S}(\mathbb{R})$. Notice that its decay property shows that differentiation and multiplication of a polynomial are closed operations inside $\mathcal{S}(\mathbb{R})$.

Proposition 8.2. *Under the Fourier transform, the following correspondence holds for* $f \in \mathcal{S}(\mathbb{R})$,

(a) $f(x+h) \longrightarrow \widehat{f}(\xi)e^{ih\xi},\ h \in \mathbb{R}$.

(b) $f(x)e^{-ixh} \longrightarrow \widehat{f}(\xi+h)$.

(c) $f(\delta x) \longrightarrow \delta^{-1}\widehat{f}(\delta^{-1}\xi),\ \delta > 0$.

(d) $f'(x) \longrightarrow i\xi\ \widehat{f}(\xi)$.

(e) $-ixf(x) \longrightarrow (d\widehat{f}/d\xi)(\xi)$.

Proof. (a)-(c) are straightforward. To prove (d) we start with definition

$$\begin{aligned}
\widehat{f'}(\xi) &= \int_{-\infty}^{\infty} f'(x)e^{-ix\xi}dx \\
&= \lim_{M\to\infty} \int_{-M}^{M} f'(x)e^{-ix\xi}dx \\
&= \lim_{M\to\infty} f(x)e^{-ix\xi}\Big|_{-M}^{M} + i\xi \int_{-M}^{M} f(x)e^{-ix\xi}dx \\
&= i\xi \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx \\
&= i\xi\widehat{f}(\xi)\ .
\end{aligned}$$

To prove (e), a formal proof is straightforward. To do a rigorous one, we need to start with the definition of differentiation

$$\begin{aligned}
&\left|\frac{\widehat{f}(\xi+h)-\widehat{f}(x)}{h} - (\widehat{-ixf})(\xi)\right| \\
= &\int_{-\infty}^{\infty} f(x)e^{-ix\xi}\left(\frac{e^{-ixh}-1}{h} + ix\right)dx\ .
\end{aligned}$$

We have

$$\frac{e^{-ixh}-1}{h}+ix=\frac{\cos xh-1}{h}-i\left(\frac{\sin xh}{h}-x\right) .$$

Using $|\cos xh-1|\le (xh)^2/2$ and $|\sin xh/h|\le |x|$ and the integrability of $xf(x)$ we could fix a large M such that

$$\left|\int_{|x|\ge M} f(x)e^{-ix\xi}\left(\frac{e^{-ixh}-1}{h}+ix\right)dx\right|<\frac{\varepsilon}{2} .$$

Now, as

$$\lim_{h\to 0}\frac{\cos xh-1}{h}=0 \ , \ \text{and} \ \lim_{h\to 0}\frac{\sin xh}{h}=x \ ,$$

uniformly on $[-M,M]$, we can find some $\delta>0$ such that

$$\left|\frac{\cos xh-1}{h}-i\left(\frac{\sin xh}{h}-x\right)\right|<\frac{\varepsilon}{2K} \ , \quad \forall x\in[-M,M] \ ,$$

where

$$K=\int_{-M}^{M}|f(x)|dx .$$

$$\left|\int_{|x|\le M} f(x)e^{-ix\xi}\left(\frac{e^{-ixh}-1}{h}+ix\right)dx\right|<K\times\frac{\varepsilon}{2K}=\frac{\varepsilon}{2} .$$

Hence we conclude

$$\left|\frac{\widehat{f}(\xi+h)-\widehat{f}(x)}{h}-(\widehat{-ixf})(\xi)\right|<\varepsilon \ , \quad |h|<\delta .$$

□

From this proposition, especially (d) and (e), we deduce

Corollary 8.3. *The Fourier transform maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$.*

Everything looks fine so far. But, nothing is perfect. When it comes to computation, we find that usually Fourier transform is difficult to carry out. This is partly because most elementary functions, such as polynomials and trigonometric functions, do not decay to infinity. Here we only give one example.

Let's compute the Fourier transform of the function $\exp(-ax^2)$, $a>0$. We need to evaluate the integral

$$\int_{-\infty}^{\infty} e^{-ax^2}e^{-ix\xi}dx.$$

By completing square,

$$\begin{aligned} -ax^2 - ix\xi &= -a\left(x^2 + \frac{i\xi x}{a}\right) \\ &= -a\left(x + \frac{i\xi}{2a}\right)^2 - \frac{\xi^2}{4a}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-ax^2 - ix\xi} dx \\ = \;& e^{-\frac{\xi^2}{4a}} \int e^{-a(x+\frac{i\xi}{2a})^2} dx \\ = \;& \sqrt{\frac{1}{a}} e^{-\frac{\xi^2}{4a}} \int e^{-y^2} dy \\ = \;& \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}} \ , \end{aligned}$$

after using the formula

$$\int e^{-y^2} dy = \pi^{1/2} \ .$$

For simplicity we have dropped the lower and upper limits in the above integrals. We conclude that

$$\mathscr{F}(e^{-ax^2})(\xi) = \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}}.$$

In particular,

$$\mathscr{F}(e^{-x^2/2})(\xi) = \sqrt{2\pi} e^{-\xi^2/2}.$$

Using the Fourier transform of $e^{-x^2/2}$ in a tricky way, one can establish the following inversion theorem.

Theorem 8.4. *The Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ one-to-one onto itself with inverse map given by the inverse Fourier transform $\mathcal{G}$ in (1).*

See, p. 139-142 in [SS] for details.

2. Derivation of The Inverse Transform We conclude this lecture with a formal derivation of the inversion theorem. It is included here for optional reading.

Let's take f to be a function on the real line, so nice that it is smooth and vanishes outside some bounded interval. For each sufficiently large l that f is equal to zero outside $[-l, l]$ we may extend the restriction of f on $[-l, l]$ to be a $2l$-periodic function in $\mathbb{R}$.

Denoting this extension still by f, we have the Fourier expansion:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right), \qquad x \in [-l, l] \\ &= \frac{1}{2l} \int_{-l}^{l} f(y)dy + \sum_{k=1}^{\infty} \left(\frac{1}{l} \int_{-l}^{l} f(y) \cos \frac{k\pi y}{l} \cos \frac{k\pi x}{l} dy + \frac{1}{l} \int_{-l}^{l} f(y) \sin \frac{k\pi y}{l} \sin \frac{k\pi x}{l} dy \right) \\ &= \frac{1}{2l} \int_{-l}^{l} f(y)dy + \sum_{k=1}^{\infty} \frac{1}{l} \int_{-l}^{l} f(y) \cos \frac{k\pi}{l} (y - x) dy \\ &= \frac{1}{2l} \int_{-\infty}^{\infty} f(y)dy + \sum_{k=1}^{\infty} \frac{1}{l} \int_{-\infty}^{\infty} f(y) \cos \frac{k\pi}{l} (y - x) dy. \end{aligned} \tag{2}$$

The last line holds because f vanishes outside $[-l, l]$. On the other hand, the function

$$\phi(\xi) = \int_{-\infty}^{\infty} f(y) \cos \xi (y - x) dy$$

is well-defined on $\mathbb{R}$. We consider its integration over $[0, \infty)$, i.e.,

$$\int_0^{\infty} \phi(\xi) d\xi = \int_0^{\infty} \int_{-\infty}^{\infty} f(y) \cos \xi (y - x) dy d\xi,$$

where x is fixed. This integral can be approximated by its Riemann sums. More precisely, we partition $\mathbb{R}$ into subintervals $[0, \pi/l]$, $[\pi/l, 2\pi/l], \cdots$, and pick $\xi_k = k\pi/l$ to form

$$\sum_{k=1}^{\infty} \phi(\frac{k\pi}{l}) \frac{\pi}{l} = \sum_{k=1}^{\infty} \frac{\pi}{l} \int_{-\infty}^{\infty} f(y) \cos \frac{k\pi}{l} (y - x) dy.$$

We see that

$$\sum_{k=1}^{\infty} \frac{\pi}{l} \int_{-\infty}^{\infty} f(y) \cos \frac{k\pi}{l} (y - x) dy$$

tends to

$$\int_0^{\infty} \int_{-\infty}^{\infty} f(y) \cos \xi (y - x) dy,$$

as $l \to \infty$. As the cosine function is even, this limit can be expressed as

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) \cos \xi (x - y) dy d\xi.$$

Using the oddity of the sine function, it can further be written as

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{i\xi(x-y)} dy d\xi.$$

By taking $l \to \infty$ in (2), we get the identity

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\xi(x-y)} f(y) dy d\xi,$$

We break up this identity into two parts. For any function f on the real line, define its Fourier transform to be

$$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy,$$

and, for any g, define its inverse Fourier transform to be

$$\mathcal{G}g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) e^{i\xi x} d\xi.$$

The Fourier and inverse Fourier transforms make sense as long as f and g decay fast at infinity so that the improper integrals are finite.